CHARACTERIZATION OF A GRAPH VALUED FUNCTION DEFINED ON BLOCKS, LINES AND CUT VERTICES OF A GRAPH

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ABSTRACT

In this paper, we introduce a new graph valued function called “Total Block Line Cutvertex graph, denoted as TBn (G) of a graph G. We present some characterizations of graphs whose TBn (G) are planar, outerplanar, maximal outerplanar and crossing number one. In addition, we establish the result of Eulerian Total Block Line Cut vertex graphs.

KEYWORDS: Planar And Nonplanar Graphs, Cutvertex, Line Graph, Wheel Graph, Total Blict Graph

INTRODUCTION

All graphs considered here are finite, undirected and without loops or multiple edges. The edges, cut vertices and blocks of a graph G are called its members. Two blocks Bᵢ and Bⱼ are adjacent if they have common cutvertex.

Definition 1.1 The Edge degree of an edge uv in G is the number of the edges adjacent to edge uv or deg u + deg v - 2. A Block vertex is a vertex in TBn (G) corresponding to a block of G.

Definition 1.2 A graph is said to be Planar if it can be embedded in a plane so that no two edges intersect. Otherwise, the graph is nonplanar.

A maximal planar graph is one to which no edge can be added without losing planarity. The concept of outerplanar graphs was studied by Tang [27]. A planar graph is said to be outerplanar if it can be embedded in a plane so that all its vertices lie on the same region. Otherwise the graph is nonouterplanar. An outerplanar graph G is maximal outerplanar if no edge can be added without losing outerplanarity. Chartrand and Harary [2] obtained a characterization of outerplanar graphs in terms of forbidden subgraphs.

Definition 1.3 The concept of non-zero inner vertex number of a planar graph was introduced by Kulli [11]. A nonnegative integer r such that any plane embedding of a planar graph G has at least r vertices not lying on the boundary of the exterior region of G is called the inner vertex number.
of G, denoted as \( i(G) \) and this indicates that G has \( r \) inner vertices. In general, the planar graphs having \( i(G) = r, \quad r > 0 \), are called r-nonouterplanar graphs. In particular, zero nonouterplanar graphs are outerplanar graphs. 1-nonouterplanar graphs will be called minimally nonouter planar graphs. For these graphs \( i(G) = 1 \). This concept has been extensively studied by Kulli [11] and others.

**Definition 1.4** The Line graph of a graph G, denoted \( L(G) \), is the graph whose vertices are the edges of G, with two vertices of \( L(G) \) adjacent whenever the corresponding edges of G are adjacent. The concept of the Line graph of a given graph is so natural that it has been independently discovered by many authors giving different name.

**Definition 1.5** The crossing number \( C(G) \) of a graph G is the minimum number of pair wise intersections (or crossings) of its edges when G is drawn in the plane. Obviously, \( C(G) = 0 \) if and only if G is planar. If \( C(G) = 1 \), then G is said to have crossing number one.

**Definition 1.6** A vertex v of G is called a cut vertex if its removal produces a disconnected graph. That is, \( G-v \) has at least two components.

**Definition 1.7** A Wheel graph \( W_n \) is a graph with n vertices formed by connecting a single vertex to all vertices of an \((n-1)\) cycle.

All undefined terms may be referred to Harary [8].

We need the following theorems for the proof of our further results.

**Theorem 1.1**[8]: If G is a graph \((V,E)\) whose vertices have degree \( d_i \), then Line graph \( L(G) \) has \( E \) vertices and \( E_L \) edges, where \( E_L = -E + \frac{1}{2} \sum d_i^2 \).

**Theorem 1.2**[25]: The line graph \( L(G) \) of a graph G is planar if and only if G is planar, the degree of each vertex of G is atmost 4 and every vertex of degree 4 is a cutvertex.

**Theorem 1.3**[4]: The Line graph \( L(G) \) of graph G is outerplanar if and only if the degree of each vertex of G is atmost 3 and every vertex of degree 3 is a cutvertex.

**Theorem 1.4**[12]: The Line graph of G has crossing number one if and only if G is planar and (i) or (ii) holds.

(i) The maximum degree \( \Delta(G) \) is 4 and there is a unique non-cutvertex of degree 4.
(ii) The maximum degree $\Delta(G)$ is 5, every vertex of degree 4 is a cut vertex, there is a unique vertex of degree 5 and it has atmost 3 edges in any block.

**Theorem 1.5[8]:** A graph $G(V,E)$ is planar if and only if $|E| \leq 3|V| - 6$.

**Theorem 1.6[8]:** If $G$ is a nontrivial connected graph with $V$ vertices which is not a path, then $l^*(G)$ is Hamiltonian for all $n \geq |V| - 3$.

### MAIN RESULTS

**Definition 2.1** Total Blict graph $TB_n(G)$ of a graph $G$ is the graph whose vertex set is the union of the set of edges, set of cut vertices and set of blocks of $G$ in which two vertices are adjacent if and only if the corresponding members of $G$ are adjacent or incident except the adjacency of cut vertices. In Figure 1.1, a graph $G$ and its Total Blict graph $TB_n(G)$ are shown.

**Remark 2.1:** For any graph $G$, $L(G) \subset TB_n(G)$.

**Remark 2.2:** For any cycle $C_v$, $V \geq 3$, $i[TB_n(G)] \geq 1$.

In particular $i[TB_n(C_3)] = 1$.

**Remark 2.3:** For every non-separable graph $G$, $TB_n(G)$ is a block.

**Remark 2.4:** Every bridge in $G$ forms a pendant edge in $TB_n(G)$.

**Remark 2.5:** For any non-separable graph $G$ an edge ‘a’ with edge degree odd corresponds to the vertex ‘a’ in $TB_n(G)$ whose vertex degree is even and vice versa.

**Remark 2.6:** For any separable graph $G$ an edge ‘a’ incident to the cutvertex corresponds to the vertex ‘a’ of odd degree in $TB_n(G)$.

**Remark 2.7:** For any graph $G$, $TB_n(G)$ is a bridgeless graph.
Theorem 2.1: For any nontrivial connected \((V, E)\) graph \(G\) whose vertices have degree \(d_i\), \(C\) is the number of the cutvertices in \(G\), \(B_k\) be the number of blocks then \(TB_n(G)\) has \((E + B_k + C)\) vertices and \(\frac{1}{2} \sum_{j=1}^{n} d_j^2 + \sum_{j=1}^{r} \deg C_j + \sum_{i=1,i\neq j}^{k} B_{i,j}\) edges, where \(C_j\) is the \(j^{th}\) cutvertex. \(B_{i,j}\) denotes that \(B_i\) is adjacent to \(B_j\).
Proof: By the definition of $TB_n(G)$, the number of vertices is $(E + B_k + C)$. For the number of edges, since $L(G) \subseteq TB_n(G)$, by Theorem 1.1[8], $-E + \frac{1}{2} \sum d_i^2$ edges are contributed to $TB_n(n)$. By definition, every block vertex is adjacent to vertices corresponding to edges from which it is formed in $G$. This gives $E$ edges to $TB_n(G)$. Every cutvertex is adjacent to the vertices corresponding to the edges incident to it in $G$. This adds $\sum_{j=1}^{c} \deg C_j$ edges to $TB_n(G)$. These blocks $B_j$ adjacent to $B_j$ for $i \neq j$ gives $\sum_{i,j=1,i \neq j}^{k} B_{i,j}$ edges this adds to the total number of edges to $TB_n(G)$.

Hence the number of edges in $TB_n(G)$ is given by

$$E[TB_n(G)] = -E + \frac{1}{2} \sum_{i=1}^{c} d_i^2 + E + \sum_{j=1}^{c} \deg C_j + \sum_{i,j=1,i \neq j}^{k} B_{i,j}$$

Hence the proof.

In the following theorem we establish the planarity of $TB_n(G)$.

Theorem 2.2: The Total Blict graph $TB_n(G)$ of graph $G$ is planar if and only if $\Delta(G) \leq 3$ and every vertex of degree 3 is a cut vertex.

Proof: Suppose $TB_n(G)$ is planar. Assume $\Delta(G) > 3$. Let $v$ be a vertex of degree 4 in $G$, we have the following cases.

Case 1: If $v$ is a non cutvertex, then the number of edges incident to $v$ forms $<K_4>$ as a subgraph in $L(G)$. By definition of $TB_n(G)$ the block vertex is adjacent to all the vertices of $<K_4>$ which gives $<K_4> \subseteq TB_n(G)$ which is non planar, a contradiction.

Case 2: If $v$ is a cutvertex then the number of edges incident to $v$ forms $<K_4>$ as a subgraph in $L(G)$. By the definition, the cutvertex $V$ is adjacent to each vertex of $<K_4>$ gives $<K_4> \subseteq TB_n(G)$, a contradiction for planarity of $TB_n(G)$. 
Suppose $\Delta(G) \leq 3$ and $G$ has a non-cutvertex $v$ of degree 3. Clearly $v$ lies on exactly one block. Then by Theorem 1.3[4], $i[L(G)] \geq 1$. Let $B_k$ be the block vertex belonging to the block where $v$ lies in $TBn(G)$. $B_k$ is adjacent to all the vertex of $L(G)$, which gives at least one crossing and the adjacencies of blocks gives more crossings. Hence $TBn(G)$ is nonplanar, a contradiction.

Conversely, suppose $\Delta(G) \leq 3$ and every vertex of degree 3 is a cutvertex. We have the following cases.

**Case 1:** If every cutvertex of the degree 3 lies on 3 blocks of $G$, then clearly $G$ is a tree. Each block of $TBn(G) \cap \{B_k\}$ is either $K_3$ or $K_4$. Since each block of $G$ is an edge, each vertex of $TBn(G) \cap \{B_k + \sum C_j\}$ is incident with an end edge gives each block of $TBn(G)$ either $K_2$ or $K_3$ or $K_4$. The adjacent of blocks in $TBn(G)$ gives $K_3$ as subgraphs. Hence $TBn(G)$ is planar.

**Case 2:** If every cut vertex of degree 3 lies on 2 blocks, then every block of $G$ is either $K_2$ or $C_v$. $v \geq 3$. In $TBn(G)$ $(E + B_k + C_j)^3 - 6 \geq \frac{1}{2} \sum_{i=1}^{k} d_i^2 + \sum_{j=1}^{m} \deg C_j + \sum_{i,j=1,i \neq j}^{k} B_{i,j}$ by theorem 1.5[8], $TBn(G)$ is planar. Hence the theorem is proved.

In the next theorem we obtain a condition for the Total Blict graph to be outer planar.

**Theorem 2.3:** The Total Blict graph $TBn(G)$ of a graph $G$ is outerplanar if and only if $G$ is a path.

Proof: Suppose $TBn(G)$ is outerplanar. Assume $G$ has a vertex $v$ of degree 3. We consider the following cases.

**Case 1:** If $v$ is a cutvertex and lies on 3 blocks, then $K_{1,3}$ is a subgraph of $G$ and $L[K_{1,3}] = K_3$. In $TBn(G)$ vertex $v$ is adjacent to all the vertices of $K_3$ forming $<K_3>$ as a subgraph which is nonouter planar, a contradiction. Hence $G$ has no vertex of degree 3.

**Case 2:** Suppose $G$ is a cycle $C_v$ where $v \geq 3$, for $v = 3$ by definition, $TBn(G)$ contains $K_4$ which is nonouter planar. From case 1 and case 2, $G$ should be a path.

Conversely, If $G$ is a path, then $TBn(G)$ is outerplanar.
If $G$ is $P_1$, then $TB_n(G)$ is also $P_1$. If $G$ is $P_2$ then $TB_n(G)$ is $Cr_5$ which is outerplanar. For every addition of an edge to $P_2$, we get an addition of $Cr_5-X$ to $TB_n(G)$ of $P_2$, where $X$ is the common edge in $TB_n(G)$, which is outer planar. In general for every addition of an edge to the path gives $(n - 1)$ times $Cr_5-X$ in $TB_n(G)$. Hence it is outer planar.

**Theorem 2.4:** For any graph $G$ with $P > 2$ vertices, the Total Blict graph $TB_n(G)$ is not maximal outerplanar.

**Proof:** We prove the theorem by two cases,

**Case 1:** If $G$ consists a cycle $C_3$, Then $TB_n(G) = K_4$ which is non-outerplanar. There is nothing to discuss further.

If $G$ consists a cycle $C_p$, $p > 3$, the vertex of $TB_n(G)$ corresponding to the block in $G$ spoils the outer planarity of $TB_n(G)$.

**Case 2:** If $G$ consists a path $P_2$, then $TB_n(G)$ consists $C_4$ which is a non-maximal outer planar.

**Theorem 2.5:** The Total Blict graph $TB_n(G)$ of a graph $G$ is minimally nonouter planar if and only if $G$ is a cycle $C_3$. **Proof:** Suppose Total Blict graph $TB_n(G)$ is minimally nonouter planar. Assume $\Delta(G) \geq 3$. We consider the following cases.

**Case 1:** If $\Delta(G) > 3$, then by theorem 2.2 $TB_n(G)$ is nonplanar, a contradiction.

**Case 2:** If $\Delta(G) = 3$ we have the following subcases.

*Subcase (i):* If $G$ is a tree and has more than one vertex of degree 3. Then $G$ contains more then one $K_{1,3}$ as subgraph. Each $K_{1,3}$ in $G$ gives $K_3$ in $L(G)$ and adjacency of blocks and edges incident to cutvertices gives a graph which contains $K_4$ in $TB_n(G)$. Hence $i[TB_n(G)] > 1$, a contradiction.

*Subcase (ii):* If $G$ is not a tree and has more than one vertex of degree 3. Then each vertex of degree 3 in $G$ gives a subgraph $<K_4>$ in $TB_n(G)$. Adjacency of blocks with the edges and itself gives $i[TB_n(G)] > 1$, a contradiction. 

*Subcase (iii):* If $G$ is not a tree, has cutvertex $v$ of degree 3 which lies on 2 blocks of $G$. Clearly one block is $K_2$ and other is $C_v (v \geq 3)$. Let $e_1, e_2, e_3$ be the
edges incident on \( v \) and \( e_1, e_2, e_3 \in \mathcal{C}_v \). In \( TBn \) \((G)\) \( e_1, e_2, e_3 \) together with \( v \) form \( K_4 \), along with adjacency of blocks gives non outer planar graph with \( i[\mathcal{TBn}(G)]>1 \), a contradiction.

Conversely, Suppose \( G \) is a cycle \( C_3 \) then by Remark 2.2, \( i[\mathcal{TBn}(G)]=1 \). Hence \( TBn \) \((G)\) is minimally nonouterplanar.

**Theorem 2.6:** The Total Blict graph \( TBn \) \((G)\) of a graph \( G \) has crossing number one if and only if \( G \) is planar and \( \Delta(G) \leq 3 \) for every vertex \( v \) of \( G \), \( G \) has exactly two adjacent non cutvertices of degree 3.

**Proof:** Suppose \( G \) is planar and \( \Delta(G) \leq 3 \) for every vertex \( v \) of \( G \), \( G \) has exactly two adjacent non cut vertices of degree 3. Then \( G \) has a block homeomorphic to \( K_4 - X \) or \( G \) has a block \( K_4 - X \) as a sub graph. In each case, let \( e = uv \) be an edge incident on two adjacent non cut vertices of degree 3 which lies in the interior region of \( K_4 - X \).

In \( L(G) \), \( L(K_4 - X) \) gives a block homeomorphic to \( w_5 \) or a block \( w_3 \) as a sub graph in \( L(G) \). In \( TBn \) \((G)\) the adjacency of inner vertex of \( w_5 \) and vertex corresponding to block gives one crossing. Hence \( c[\mathcal{TBn}(G)]=1 \)

Conversely, if \( TBn \) \((G)\) has crossing number one then \( G \) is planar.

**Case 1:** Suppose \( G \) has crossing number one. Let \( \Delta(G) > 3 \). By Theorem 2.2, \( TBn \) \((G)\) is nonplanar, a contradiction. Therefore \( \Delta(G) \leq 3 \).

**Case 2:** If \( G \) has atleast two cutvertices of degree 3.

Let \( v_1 \) and \( v_2 \) be the two cut vertices of degree 3 incident on \( e_1, e_2, e_3 \) and \( f_1, f_2, f_3 \) edges respectively. Hence \( L(G) \) has two induced sub graphs as \( C_3 \). Since \( v_1 \) is incident on \( e_1, e_2, e_3 \) and \( v_2 \) is incident on \( f_1, f_2, f_3 \) by definition of \( TBn \) \((G)\), \( v_1 \) is adjacent to each vertex of \( C_3 \) which gives \( K_4 - X \) as a sub graph in \( TBn \) \((G)\). Adjacency of block vertex with \( K_4 - X \) gives \( K_5 - X \) as a sub graph in \( TBn \) \((G)\). Hence \( TBn \) \((G)\) has two induced sub graphs as \( K_5 - X \). Clearly, \( c[\mathcal{TBn}(G)]>1 \), a contradiction. Hence \( G \) has only one cut vertex of degree 3.
Suppose TBn (G) has crossing number one and exactly one cut vertex of degree 3. Then TBn (G) contains $\kappa_4$ as a sub graph and G must contain a $K_{1,3}$ as a sub graph which is planar.

Hence the proof.

**Theorem 2.7:** For any graph G, TBn (G) is non Eulerian.

**Proof:** Proof of this theorem is obvious by Remark 2.5 and Remark 2.6.

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**REFERENCES**


